

Energy functionals and Kähler-Ricci solitons

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Abstract In this paper, we generalize Chen-Tian energy functionals to Kähler-Ricci solitons and prove that the properness of these functionals is equivalent to the existence of Kähler-Ricci solitons. We also discuss the equivalence of the lower boundedness of these functionals and their relation with Tian-Zhu's holomorphic invariant.

1 Introduction

In [6], a series of energy functionals $E_k (k = 0, 1, \dots, n)$ were introduced by X.X. Chen and G. Tian which were used to prove the convergence of the Kähler Ricci flow under some curvature assumptions. The first energy functional E_0 of this series is exactly the K -energy introduced by Mabuchi in [12], which can be defined for any Kähler potential $\varphi(t)$ on a Kähler manifold (M, ω) as follows:

$$\frac{d}{dt} E_0(\varphi(t)) = -\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (R_\varphi - r) \omega_\varphi^n.$$

Here R_φ is the scalar curvature with respect to the Kähler metric $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$, $r = \frac{[c_1(M)][\omega]^{n-1}}{[\omega]^n}$ is the average of R_φ and $V = [\omega]^n$ is the volume.

It is well-known that the behavior of the K -energy plays a central role on the existence of Kähler-Einstein metrics and constant scalar curvature metrics. In [1], Bando-Mabuchi proved that the K -energy is bounded from below on a Kähler-Einstein manifold with $c_1(M) > 0$. It has been shown by G. Tian in [17][18] that M admits a Kähler-Einstein metric if and only if the K -energy or F functional defined by Ding-Tian [8] is proper. Thus, it is natural to study the relation between E_k functionals and Kähler-Einstein metrics. Following a question posed by Chen in [3], Song-Weinkove studied the lower bound of energy functionals E_k on Kähler-Einstein manifolds. Shortly afterwards, N. Pali [13] gave a formula between E_1 and the K -energy E_0 , which implies E_1 has a lower bound if the K -energy is bounded from below. Inspired by Song-Weinkove and Pali's work, we proved that the lower boundedness of F functional, the K -energy and E_1 are equivalent in the canonical Kähler class in [5][10], and we proved a general formula which gives the relations of all energy functionals E_k in [9]. In [14] Y. Rubinstein extended these results and proved all the lower boundedness and properness of E_k functionals are equivalent under some natural restrictions.

For the case of Kähler-Ricci solitons, Tian-Zhu generalized the K -energy and F functional in [19] and proved that these generalized energy are bounded from below on a Kähler manifold which admits a Kähler-Ricci soliton. In [2] Cao-Tian-Zhu proved the properness of the generalized energy functionals. Inspired by these work, we will define the generalized Chen-Tian energy functionals \tilde{E}_k in Section 2 and prove the following result:

Theorem 1.1. *Let (M, ω) be a compact Kähler manifold with $c_1(M) > 0$ and $\omega \in 2\pi c_1(M)$. For any $k \in \{0, 1, \dots, n\}$ we have*

- (a) *If \tilde{E}_k is proper on $\mathcal{M}_{X,k}^+(\omega)$, then M admits a Kähler-Ricci soliton with respect to X ;*
- (b) *If M admits a Kähler-Ricci soliton ω_{KS} , then \tilde{E}_k is proper on $\mathcal{M}_G(\omega_{KS})' \cap \mathcal{M}_{X,k}^+(\omega_{KS})$.*

where $\mathcal{M}_{X,k}^+(\omega_{KS})$ and $\mathcal{M}_G(\omega_{KS})'$ are some subspaces of Kähler potentials defined in Section 2.

The idea of the proof is more or less standard. We follow the continuity method from [14] and [2] to prove this. The crucial point is that by the construction of the generalized energy functionals \tilde{E}_k , all the arguments for the Kähler-Einstein case work very well for our situation. Following the results in [5] and [10] we discuss the lower bound of these energy functionals:

Theorem 1.2. *Let (M, ω) be a compact Kähler manifold with $c_1(M) > 0$ and $\omega \in 2\pi c_1(M)$. Then for any $k \in \{0, 1, \dots, n\}$, \tilde{E}_k is bounded from below on $\mathcal{M}_{X,k}^+(\omega)$ if and only if \tilde{F} is bounded from below on $\mathcal{M}_X(\omega)$. Moreover, we have*

$$\inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') = (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{F}_\omega(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta_X} \omega^n, \quad (1.1)$$

where $u_0 = -h_\omega + \theta_X$ and $C_{\omega,X,k}$ is given by

$$C_{\omega,X,k} = \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \frac{1}{V} \int_M \sqrt{-1} \partial u_0 \wedge \bar{\partial} u_0 \wedge (\sqrt{-1} \partial \bar{\partial} u_0)^{k-1} \wedge e^{\theta_X} \omega^{n-k}. \quad (1.2)$$

Here we take the ideas from [8] to prove Theorem 1.2. For the energy functionals E_k , there are two different ways to prove their equivalence. In [5] we use the Kähler-Ricci flow to prove the equivalence of the lower boundedness of the K energy and E_1 energy, and in [10] we use Perelman's estimates to prove the equivalence of the K energy and F functional. The flow method is very tricky and we lack some crucial estimates here. In [14] Y. Rubinstein proved that the equivalence of the lower boundedness of the energy functionals E_k and F , which relies on an interesting observation on the relation of E_n and F (cf. Lemma 2.4 in [14]). Here it seems difficult to find such a relation in the case of generalized energy functionals. Fortunately, we can use the continuity method in [8] to overcome these difficulties.

As a by-product of Theorem 1.2, we have the following result:

Theorem 1.3. *Let M be a compact Kähler manifold with $c_1(M) > 0$ and ω be any given Kähler metric in $2\pi c_1(M)$. If \tilde{F} is bounded from below for the solution φ_t of the equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\omega - \theta_X(\varphi) - t\varphi}, \quad (1.3)$$

then \tilde{F} is bounded from below in the class $2\pi c_1(M)$. Moreover, we have

$$\inf_{\omega' \in 2\pi c_1(M)} \tilde{F}_\omega(\omega') = \inf_{t \in [0,1)} \tilde{F}_\omega(\varphi_t).$$

Similar results also hold for \tilde{E}_k on $\mathcal{M}_{X,k}^+(\omega)$ with $k = 0, 1, \dots, n$.

This result is inspired by the beautiful work [4]. In [4] X. X. Chen proved that if the K -energy is bounded from below and the infimum of the Calabi energy vanishes along a particular geodesic ray, then the K -energy is bounded from below in the Kähler class. As an application, he essentially proved that for any Kähler class admits constant scalar curvature metric, the K -energy in a nearby Kähler class with possibly different complex structure is bounded from below. We remark that under the assumption of Theorem 1.3, the solution φ_t of (1.3) will exist for all $t \in [0, 1)$. Theorem 1.3 shows that if the K -energy is bounded along one solution φ_t , then the K -energy is bounded from below in the whole Kähler class. It is interesting to know whether there is a similar phenomenon for the Kähler-Ricci flow:

Question 1.4. *If the K -energy is bounded from below along a certain Kähler-Ricci flow, is the K -energy bounded from below in the class $2\pi c_1(M)$?*

In [19], Tian-Zhu introduced a new holomorphic invariant $\mathcal{F}_X(\cdot)$ from the space of holomorphic vector fields $\eta(M)$ into \mathbb{C} :

$$\mathcal{F}_X(Y) = \int_M Y(h_g - \theta_X(g)) e^{\theta_X(g)} \omega_g^n, \quad Y \in \eta(M).$$

The invariant $\mathcal{F}_X(\cdot)$ is defined for any holomorphic vector fields $X \in \eta(M)$ and it is independent of the choice of g with the Kähler class $\omega_g \in 2\pi c_1(M)$. When $X = 0$, $\mathcal{F}_X(\cdot)$ is exactly the Futaki invariant. By the definition, we see that \mathcal{F}_X is an obstruction to the existence of Kähler-Ricci solitons. The next result shows that the holomorphic invariants defined by \tilde{E}_k are scalar multiples of $\mathcal{F}_X(\cdot)$, which generalized the results for energy functionals E_k (cf. [11][9]).

Theorem 1.5. *Let Y be a holomorphic vector field and $\{\Phi(t)\}_{|t|<\infty}$ the one-parameter subgroup of automorphisms induced by $\text{Re}(Y)$, we have*

$$\frac{d}{dt} \tilde{E}_k(\varphi) = \frac{(k+1)n}{V} \mathcal{F}_X(Y),$$

where φ is given by $\Phi_t^ \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$.*

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2 Energy functionals

In this section, we recall some energy functionals introduced by Tian-Zhu in [19] and give the definition of the generalized Chen-Tian energy function.

Let M be an n -dimensional compact Kähler manifold with positive first Chern class, and ω be a fixed Kähler metric in the Kähler class $2\pi c_1(M)$. Then there is a smooth real-valued function h_ω such that

$$\text{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega, \quad \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

Suppose that X is a holomorphic vector field on M so that the integral curve of K_X of the imaginary part $\text{Im}(X)$ of X consists of isometries of ω . By the Hodge decomposition theorem, there exists a unique smooth real-valued function θ_X on M such that

$$i_X\omega = \sqrt{-1}\partial\bar{\partial}\theta_X, \quad \int_M (e^{\theta_X} - 1)\omega^n = 0.$$

Now we define the space of Kähler potentials which are invariant under $\text{Im}(X)$:

$$\mathcal{M}_X(\omega) = \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \text{Im}(X)(\varphi) = 0\}.$$

Tian-Zhu in [19] introduced the following functional, which can be regarded as a generalization of Mabuchi's K energy,

$$\tilde{E}_{0,\omega}(\varphi) = \frac{n}{V} \int_0^1 \int_M \sqrt{-1}\partial\frac{\partial\varphi_t}{\partial t} \wedge \bar{\partial}(h_{\varphi_t} - \theta_X(\varphi_t)) \wedge e^{\theta_X(\varphi_t)}\omega_{\varphi_t}^{n-1} \wedge dt \quad (2.1)$$

where $\varphi_t(t \in [0, 1])$ is a path connecting 0 and φ in $\mathcal{M}_X(\omega)$ and

$$\theta_X(\varphi_t) = \theta_X + X(\varphi_t)$$

is the potential function of X with respect to the metric ω_{φ_t} . We define the following functionals on $\mathcal{M}_X(\omega)$:

$$\tilde{I}_\omega(\varphi) = \frac{1}{V} \int_M \varphi(e^{\theta_X}\omega^n - e^{\theta_X(\varphi)}\omega_\varphi^n),$$

and

$$\tilde{J}_\omega(\varphi) = \frac{1}{V} \int_0^1 \int_M \frac{\partial\varphi_t}{\partial t} (e^{\theta_X}\omega^n - e^{\theta_X(\varphi_t)}\omega_{\varphi_t}^n) \wedge dt.$$

As before, $\varphi_t(t \in [0, 1])$ is a path connecting 0 and φ in $\mathcal{M}_X(\omega)$. Then for any path φ_t , we have

$$\frac{d}{dt}(\tilde{I}_\omega(\varphi_t) - \tilde{J}_\omega(\varphi_t)) = -\frac{1}{V} \int_M \frac{\partial\varphi}{\partial t} (\Delta_{\varphi_t} + X)\varphi_t e^{\theta_X(\varphi_t)}\omega_{\varphi_t}^n.$$

Then, by Lemma 3.1 in [2] there exist two positive constants $c_1(n)$ and $c_2(n)$ such that

$$c_1 I_\omega(\varphi) \leq \tilde{I}_\omega(\varphi) - \tilde{J}_\omega(\varphi) \leq c_2 I_\omega(\varphi), \quad (2.2)$$

where

$$I_\omega(\varphi) = \frac{1}{V} \int_M \varphi(\omega^n - \omega_\varphi^n). \quad (2.3)$$

In [19] Tian-Zhu defined the generalized F functional which is defined by Ding-Tian in [8] as follows

$$\tilde{F}_\omega(\varphi) = \tilde{J}_\omega(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta_X} \omega^n - \log \left(\frac{1}{V} \int_M e^{h_\omega - \varphi} \omega^n \right). \quad (2.4)$$

The generalized F functional has exactly the same behavior as in the Kähler-Einstein case. For example, the generalized K -energy and \tilde{F} functional are related by the identity (cf. [19])

$$\tilde{E}_0(\varphi) = \tilde{F}(\varphi) + \frac{1}{V} \int_M u e^{\theta_X(\varphi)} \omega_\varphi^n - \frac{1}{V} \int_M u_0 e^{\theta_X} \omega^n + \log \left(\frac{1}{V} \int_M e^{h_\omega - \varphi} \omega^n \right), \quad (2.5)$$

where u is defined by

$$u(\varphi) = -h_\varphi + \theta_X(\varphi) = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega + \theta_X(\varphi), \quad (2.6)$$

and $u_0 = -h_\omega + \theta_X$. It follows that \tilde{E}_0 is always bigger than \tilde{F} up to a constant:

$$\tilde{E}_0(\varphi) \geq \tilde{F}(\varphi) - \frac{1}{V} \int_M u_0 e^{\theta_X} \omega^n. \quad (2.7)$$

Now we recall some results in [2]. Let $K_0(\supset K_X)$ be a maximum compact subgroup of the automorphisms group of M such that $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\eta \in K_0$ and any $\sigma \in K_X$. If ω_{KS} is a Kähler-Ricci soliton with respect to the holomorphic vector field X , we define the inner product by

$$(\varphi, \psi) = \int_M \varphi \psi e^{\theta_X(\omega_{KS})} \omega_{KS}^n,$$

and denote by

$$\Lambda_1(\omega_{KS}) = \{u \in C^\infty \mid \Delta_{KS} u + X(u) = -u\}.$$

For any compact subgroup $G \supset K_X$ of K_0 with $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\eta \in G$ and any $\sigma \in K_X$, we denote by $\mathcal{M}_G(\omega_{KS})'$ the space of G -invariant Kähler potentials perpendicular to $\Lambda_1(\omega_{KS})$. We call a functional $F(\varphi)$ proper on $\mathcal{M}_X(\omega)$, if there exists an increasing function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\lim_{t \rightarrow +\infty} \rho(t) = +\infty$ such that for any $\varphi \in \mathcal{M}_X(\omega)$, $F(\varphi) \geq \rho(I_\omega(\varphi))$, where $I_\omega(\varphi)$ is given by (2.3).

In [2], Cao-Tian-Zhu proved the following result, which is crucial in the proof of Theorem 1.1.

Theorem 2.1. (cf. [2]) *If M admits a Kähler-Ricci soliton, then \tilde{F} is proper on $\mathcal{M}_G(\omega_{KS})'$.*

Inspired by the work in [19] and [9], we define the generalized Chen-Tian energy functionals \tilde{E}_k as follows:

Definition 2.2. We define the generalized Chen-Tian energy functionals for any $k = 1, 2, \dots, n$,

$$\tilde{E}_{k,\omega}(\varphi) = \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi) + (k+1) \tilde{E}_0(\varphi),$$

where

$$\begin{aligned} \tilde{G}_{k,\omega}(\varphi) &= -\frac{1}{V} \int_M \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \\ &\quad + \frac{1}{V} \int_M \sqrt{-1} \partial u_0 \wedge \bar{\partial} u_0 \wedge (\sqrt{-1} \partial \bar{\partial} u_0)^{k-1} \wedge e^{\theta_X} \omega^{n-k}, \end{aligned}$$

where $u = -h_\varphi + \theta_X(\varphi)$ and $u_0 = -h_\omega + \theta_X$.

Remark 2.3. For Chen-Tian energy functionals E_k , there are many different expressions as in [6] [15] and [14]. It is interesting how to write the generalized functionals \tilde{E}_k as similar expressions.

By the definition, it is easy to check that all of \tilde{E}_k satisfy the following cocycle condition

$$\tilde{E}_{k,\omega}(\varphi) + \tilde{E}_{k,\omega_\varphi}(\psi - \varphi) = \tilde{E}_{k,\omega}(\psi),$$

for any $\varphi, \psi \in \mathcal{M}_X(\omega)$. Let $k = 1$, we have the generalized Pali's formula:

$$\begin{aligned} \tilde{E}_1(\varphi) &= 2\tilde{E}_0(\varphi) + \frac{1}{V} \int_M \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-1} - \frac{1}{V} \int_M \sqrt{-1} \partial u_0 \wedge \bar{\partial} u_0 \wedge e^{\theta_X} \omega^{n-1} \\ &\geq 2\tilde{E}_0(\varphi) - C_{\omega,X}. \end{aligned} \tag{2.8}$$

Now we define the subspace of Kähler potentials for $k = 2, 3, \dots, n$

$$\mathcal{M}_{X,k}^+(\omega) = \{\varphi \in \mathcal{M}_X(\omega) \mid Ric_\varphi - L_X \omega_\varphi \geq -\frac{2}{k-1} \omega_\varphi\}, \tag{2.9}$$

and let $\mathcal{M}_{X,0}^+(\omega) = \mathcal{M}_{X,1}^+(\omega) = \mathcal{M}_X(\omega)$. The definition of (2.9) is inspired by the result in [9].

With these notations, we have the result:

Lemma 2.4. For any $\omega_\varphi \in \mathcal{M}_{X,k}^+(\omega)$ ($k \geq 2$), we have

$$\tilde{E}_k(\varphi) \geq (k+1) \tilde{E}_0(\varphi) - C_{\omega,X,k},$$

where $C_{\omega,X,k}$ is given by (1.2).

Proof. The argument is the same as in [9] and here we give the details for completeness. By the definition of \tilde{E}_k , we have

$$\begin{aligned} &\tilde{E}_k - (k+1) \tilde{E}_0 \\ &= \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i} \\ &= \frac{1}{V} \int_M \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left(\sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k+1}{i} (\sqrt{-1} \partial \bar{\partial} u)^{k-i-1} \wedge \omega_\varphi^i \right) \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} + C_{\omega,X,k} \\ &= \frac{1}{V} \int_M \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left(\sum_{i=0}^{k-1} \binom{k+1}{i} (Ric_\varphi - L_X \omega_\varphi - \omega_\varphi)^{k-i-1} \wedge \omega_\varphi^i \right) \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} + C_{\omega,X,k}. \end{aligned}$$

Observe that

$$\sum_{i=0}^{k-1} \binom{k+1}{i} (Ric_\varphi - L_X \omega_\varphi - \omega_\varphi)^{k-i-1} \wedge \omega_\varphi^i = \sum_{i=1}^k i (Ric_\varphi - L_X \omega_\varphi)^{k-i} \wedge \omega_\varphi^{i-1}. \quad (2.10)$$

For $k \geq 2$, let

$$P(x) = \sum_{i=1}^k i x^{k-i} = \left(x + \frac{2}{k-1}\right)^{k-1} + \sum_{i=2}^k a_i \left(x + \frac{2}{k-1}\right)^{k-i},$$

where a_i are the constants defined by

$$a_i = \frac{1}{(k-i)!} P^{(k-i)}\left(-\frac{2}{k-1}\right).$$

By Lemma A.1 in the appendix of [9], $a_i \geq 0$. Thus, for any $\varphi \in \mathcal{M}_{X,k}^+(\omega)$ we have

$$\begin{aligned} & \sum_{i=1}^k i (Ric_\varphi - L_X \omega_\varphi)^{k-i} \wedge \omega_\varphi^{i-1} \\ &= \left(Ric_\varphi - L_X \omega_\varphi + \frac{2}{k-1} \omega_\varphi\right)^{k-1} + \sum_{i=2}^k a_i \left(Ric_\varphi - L_X \omega_\varphi + \frac{2}{k-1} \omega_\varphi\right)^{k-i} \wedge \omega_\varphi^{i-1} \geq 0. \end{aligned}$$

Therefore, $\tilde{E}_k \geq (k+1)\tilde{E}_0 + C_{\omega,X,k}$ and the lemma is proved. □

3 Proof of Theorem 1.1

Suppose that M admits a Kähler-Ricci soliton in the class $\mathcal{M}_X(\omega)$. This implies that \tilde{F} functional is proper on $\mathcal{M}_G(\omega_{KS})'$ by Theorem 2.1, and also \tilde{E}_0 is proper on $\mathcal{M}_G(\omega_{KS})'$ by (2.7). Thus, \tilde{E}_1 is also proper on $\mathcal{M}_G(\omega_{KS})'$ by (2.8) and so is \tilde{E}_k on $\mathcal{M}_{X,k}^+(\omega) \cap \mathcal{M}_G(\omega_{KS})'$ for any $k \in \{2, \dots, n\}$ by Lemma 2.4. Thus, part (b) of Theorem 1.1 is proved.

To finish part (a) of Theorem 1.1, it suffices to prove:

Lemma 3.1. *If \tilde{E}_k is proper on $\mathcal{M}_{X,k}^+(\omega)$ for any $k \in \{2, 3, \dots, n\}$, then there exists a Kähler-Ricci soliton on M .*

Proof. We consider the complex Monge-Ampère equations with parameter $t \in [0, 1]$

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{h - \theta_X(\varphi) - t\varphi} \omega^n. \quad (3.1)$$

There exists a unique solution at $t = 0$ modulo constants by [21], and the set of $t \in [0, 1]$ such that (3.1) has a solution is open by the implicit function theorem(cf. [20]). Therefore, to prove that there is a solution for $t = 1$, it suffices to prove that $I_\omega(\varphi)$ is uniformly bounded for $0 \leq t < 1$.

Note that the solution $\varphi_t \in \mathcal{M}_{X,k}^+(\omega)$, since the equation (3.1) can be written as

$$Ric_\varphi - L_X \varphi = t\omega_\varphi + (1-t)\omega > 0.$$

Since \tilde{E}_k is proper on $\mathcal{M}_{X,k}^+(\omega)$, there exists an increasing function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\lim_{s \rightarrow +\infty} \rho(s) = +\infty$ such that $\tilde{E}_k(\varphi(t)) \geq \rho(I_\omega(\varphi(t)))$. Now we show that \tilde{E}_k is uniformly bounded from above for $t \in [0, 1)$. In fact,

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{E}_0(\varphi_t) &= \frac{n}{V} \int_M \sqrt{-1} \partial \frac{\partial \varphi}{\partial t} \wedge \bar{\partial}(h_\varphi - \theta_X(\varphi)) \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-1} \\ &= -\frac{n}{V} \int_M \sqrt{-1} \partial \frac{\partial \varphi}{\partial t} \wedge \bar{\partial} u \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-1}. \end{aligned}$$

Thus, for the solution φ_t ($0 \leq t \leq \tau \leq 1$) we have

$$\begin{aligned} \tilde{E}_0(\varphi_\tau) - \tilde{E}_0(\varphi_0) &= -\frac{n}{V} \int_0^\tau \int_M (1-t) \sqrt{-1} \partial \frac{\partial \varphi}{\partial t} \wedge \bar{\partial} \varphi \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-1} \wedge dt \\ &= \frac{1}{V} \int_0^\tau \int_M (1-t) \varphi (\Delta_\varphi + X) \frac{\partial \varphi}{\partial t} e^{\theta_X(\varphi)} \omega_\varphi^n \wedge dt \\ &= -\int_0^\tau (1-t) \frac{d}{dt} (\tilde{I} - \tilde{J}) dt \\ &= -(1-\tau)(\tilde{I} - \tilde{J})(\varphi_\tau) + (\tilde{I} - \tilde{J})(\varphi_0) - \int_0^\tau (\tilde{I} - \tilde{J}) dt \quad (3.2) \\ &\leq -c(n)(1-\tau)I_\omega(\varphi_\tau) + (\tilde{I} - \tilde{J})(\varphi_0) - \int_0^\tau (\tilde{I} - \tilde{J}) dt, \end{aligned}$$

where we have used the inequality (2.2). Hence, by the definition of \tilde{E}_k we have

$$\begin{aligned} \tilde{E}_k(\varphi_\tau) &= (k+1)E_0(\varphi_\tau) + \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_\tau) \\ &\leq (k+1)\tilde{E}_0(\varphi_0) - c(n)(k+1)(1-\tau)I_\omega(\varphi_\tau) + (k+1)(\tilde{I} - \tilde{J})(\varphi_0) \\ &\quad - (k+1) \int_0^\tau (\tilde{I} - \tilde{J}) dt + \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_\tau). \quad (3.3) \end{aligned}$$

Note that φ_τ satisfies the equation (3.1) and we have

$$\begin{aligned} u(\tau) &= \log \frac{\omega_{\varphi_\tau}^n}{\omega^n} + \varphi_\tau - h_\omega + \theta_X(\varphi_\tau) \\ &= (1-\tau)\varphi_\tau. \end{aligned}$$

Thus, we have

$$\begin{aligned}
& \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_\tau) \\
&= \frac{1}{V} \int_M \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left(\sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k+1}{i} (\sqrt{-1} \partial \bar{\partial} u)^{k-i-1} \wedge \omega_{\varphi_\tau}^i \right) \wedge e^{\theta_X(\varphi_\tau)} \omega_{\varphi_\tau}^{n-k} + C_{\omega,X,k} \\
&= \frac{1}{V} \int_M (1-\tau)^2 \sqrt{-1} \partial \varphi_\tau \wedge \bar{\partial} \varphi_\tau \wedge \\
&\quad \left(\sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k+1}{i} ((1-\tau) \sqrt{-1} \partial \bar{\partial} \varphi_\tau)^{k-i-1} \wedge \omega_{\varphi_\tau}^i \right) \wedge e^{\theta_X(\varphi_\tau)} \omega_{\varphi_\tau}^{n-k} + C_{\omega,X,k} \\
&= \frac{1}{V} \int_M (1-\tau)^2 \sqrt{-1} \partial \varphi_\tau \wedge \bar{\partial} \varphi_\tau \wedge \left(\sum_{i=1}^k i (\tau \omega_{\varphi_\tau} + (1-\tau) \omega)^{k-i} \wedge \omega_{\varphi_\tau}^{i-1} \right) \wedge e^{\theta_X(\varphi_\tau)} \omega_{\varphi_\tau}^{n-k} + C_{\omega,X,k} \\
&\leq (1-\tau)^2 c(n) I(\varphi_\tau) + C_{\omega,X,k}. \tag{3.4}
\end{aligned}$$

Combining this with inequality (3.3), for any τ sufficiently close to 1 we have

$$\rho(I(\varphi_\tau)) \leq \tilde{E}_k(\varphi_\tau) \leq C(\omega, \varphi_0).$$

Hence $I(\varphi_t)$ is uniformly bounded from above for $t \in [0, 1)$. Thus, $|\varphi_t|_{C^0}$ and all higher order estimates are uniformly bounded for any $t \in [0, 1)$ and the solution φ_t ($t \in [0, 1)$) can be extended to $t = 1$ smoothly. This concludes that M admits a Kähler-Ricci soliton. \square

4 Proof of Theorem 1.2

Suppose \tilde{F} is bounded from below on $\mathcal{M}_X(\omega)$. Then by (2.7) \tilde{E}_0 is bounded from below and so is \tilde{E}_k on $\mathcal{M}_k^+(\omega)$ by Lemma 2.4 for any $k \in \{1, 2, \dots, n\}$. Thus, it suffices to prove the following

Lemma 4.1. *If \tilde{E}_k is bounded from below on $\mathcal{M}_k^+(\omega)$ for any $k \in \{1, 2, \dots, n\}$, then \tilde{F} is bounded from below on $\mathcal{M}(\omega)$.*

Proof. For any $\psi \in \mathcal{M}_X(\omega)$, we set $\omega_s = \omega + s\sqrt{-1}\partial\bar{\partial}\psi$, and let $\varphi_{s,t}$ be the solution of the equation

$$(\omega_s + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_s - \theta_s(\varphi) - t\varphi} \omega_s^n, \tag{4.1}$$

where h_s satisfies

$$\text{Ric}(\omega_s) - \omega_s = \sqrt{-1}\partial\bar{\partial}h_s, \quad \int_M e^{h_s} \omega_s^n = V, \tag{4.2}$$

and θ_s is defined by

$$\theta_s = \theta_X + X(s\psi), \quad \int_M e^{\theta_s} \omega_s^n = V.$$

Since \tilde{E}_k is bounded from below, from the proof of Lemma 3.1 the solution $\varphi_{s,t}$ of (4.1) exists for any $t \in [0, 1)$ and each $s \in [0, 1]$. Now we have the following

Claim 4.2. *For any $s \in [0, 1]$ we have*

$$-\infty < \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t}) \leq 0. \quad (4.3)$$

Proof. By the proof of Lemma 3.1, for any $t \in [0, 1)$ we have

$$\tilde{E}_{k,\omega_s}(\varphi_{s,t}) \leq -c(n, k)(1-t)I_\omega(\varphi_{s,t}) + C(\omega_s) - (k+1) \int_0^t (\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_{\omega_s}(\varphi_{s,\tau}))d\tau.$$

By the assumption that \tilde{E}_k is bounded from below, we have

$$c(n, k)(1-t)I_\omega(\varphi_{s,t}) + (k+1) \int_0^1 (\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_{\omega_s}(\varphi_{s,\tau}))d\tau \leq C(\omega_s). \quad (4.4)$$

Note that $\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_{\omega_s}(\varphi_{s,\tau})$ is increasing with respect to τ , we have

$$0 \leq \tilde{I}_{\omega_s}(\varphi_{s,t}) - \tilde{J}_{\omega_s}(\varphi_{s,t}) \leq \frac{1}{1-t} \int_t^1 (\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_{\omega_s}(\varphi_{s,\tau}))d\tau.$$

and

$$\lim_{t \rightarrow 1^-} (1-t)(\tilde{I}_{\omega_s}(\varphi_{s,t}) - \tilde{J}_{\omega_s}(\varphi_{s,t})) = 0. \quad (4.5)$$

By Proposition 3.1 in [2], there exists two constants $c_1 = c_1(X, \omega)$ and $c_2 = c_2(X, \omega)$ such that for any $t \in [\frac{1}{2}, 1)$

$$\|\varphi_{s,t}\|_{C^0} \leq c_1 I_{\omega_s}(\varphi_{s,t}) + c_2.$$

Combining this with (4.5)(2.2), for any $s \in [0, 1]$ we have

$$\lim_{t \rightarrow 1^-} (1-t)\|\varphi_{s,t}\|_{C^0} = 0. \quad (4.6)$$

Note that

$$\frac{d}{dt} \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = \int_M (\Delta_{s,t} + X) \frac{\partial \varphi_{s,t}}{\partial t} e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = 0,$$

we infer that

$$\int_M e^{h_s - t\varphi_{s,t}} \omega_{\varphi_{s,t}}^n = \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = \int_M e^{\theta_s} \omega_s^n = V.$$

Combining this with (4.6), for any $s \in [0, 1]$ we have

$$\lim_{t \rightarrow 1^-} \int_M e^{h_s - \varphi_{s,t}} \omega_s^n = \lim_{t \rightarrow 1^-} \int_M e^{h_s - t\varphi_{s,t}} \cdot e^{(t-1)\varphi_{s,t}} \omega_s^n = V, \quad (4.7)$$

and we can infer that

$$\begin{aligned} \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t}) &= \lim_{t \rightarrow 1^-} \left(\tilde{J}_{\omega_s}(\varphi_{s,t}) - \frac{1}{V} \int_M \varphi_{s,t} e^{\theta_s} \omega_s^n \right) \\ &= - \int_0^1 (\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_{\omega_s}(\varphi_{s,\tau}))d\tau \leq 0, \end{aligned} \quad (4.8)$$

where we used (4.4) and the fact that (cf. Proposition 1.1 in [2])

$$\tilde{J}_{\omega_s}(\varphi_{s,t}) - \frac{1}{V} \int_M \varphi_{s,t} e^{\theta_s} \omega_s^n = -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_{\omega_s}(\varphi_{s,\tau})) d\tau.$$

Thus, the claim is proved. \square

Claim 4.3. *For any $s \in [0, 1]$, we have*

$$\lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t}) = \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\varphi_{0,t}).$$

In other words, the limit $\lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t})$ is independent of s .

Proof. By (4.2) we have

$$h_s = -\log \frac{\omega_s^n}{\omega^n} - s\psi + h_\omega + c_s \quad (4.9)$$

where c_s is a constant given by

$$\int_M e^{h_\omega - s\psi + c_s} \omega^n = V.$$

Thus, (4.1) can be written as

$$(\omega + \sqrt{-1} \partial \bar{\partial} (s\psi + \varphi_{s,t}))^n = e^{h_\omega - \theta_X - X(s\psi + \varphi_{s,t}) - t\varphi_{s,t} - s\psi + c_s} \omega^n.$$

Let $\hat{\varphi}_{s,t} = s\psi + \varphi_{s,t} - c_s$, we have

$$(\omega + \sqrt{-1} \partial \bar{\partial} \hat{\varphi}_{s,t})^n = e^{h_\omega - \theta_X(\hat{\varphi}_{s,t}) - t\hat{\varphi}_{s,t} - (1-t)(s\psi - c_s)} \omega^n. \quad (4.10)$$

Taking derivative with respect to s , we have

$$(\Delta_{s,t} + X) \frac{\partial \hat{\varphi}_{s,t}}{\partial s} = -t \frac{\partial \hat{\varphi}_{s,t}}{\partial s} - (1-t) \left(\psi - \frac{dc_s}{ds} \right). \quad (4.11)$$

Direct calculation shows that

$$\begin{aligned} \frac{\partial}{\partial s} \left(\tilde{J}_\omega(\hat{\varphi}_{s,t}) - \frac{1}{V} \int_M \hat{\varphi}_{s,t} e^{\theta_X} \omega^n \right) &= -\frac{1}{V} \int_M \frac{\partial \hat{\varphi}_{s,t}}{\partial s} e^{\theta_X(\hat{\varphi}_{s,t})} \omega_{s,t}^n \\ &= -\frac{1-t}{tV} \int_M \left(\psi - \frac{dc_s}{ds} \right) \omega_{s,t}^n, \end{aligned} \quad (4.12)$$

where we used (4.11). Note that for any $s \in [0, 1]$ we have

$$\lim_{t \rightarrow 1^-} \int_M e^{h_\omega - \hat{\varphi}_{s,t}} \omega^n = \lim_{t \rightarrow 1^-} \int_M e^{h_s - \varphi_{s,t}} \omega_s^n = V, \quad (4.13)$$

where we used (4.7) and (4.9). Combining (4.12) with (4.13), for any $s \in [0, 1]$ we have

$$\begin{aligned} \lim_{t \rightarrow 1^-} \left| \tilde{F}_\omega(\hat{\varphi}_{s,t}) - \tilde{F}_\omega(\hat{\varphi}_{0,t}) \right| &= \lim_{t \rightarrow 1^-} \left| \int_0^s \frac{\partial}{\partial \tau} \left(\tilde{J}_\omega(\hat{\varphi}_{\tau,t}) - \frac{1}{V} \int_M \hat{\varphi}_{\tau,t} e^{\theta_X} \omega^n \right) d\tau \right| \\ &= \lim_{t \rightarrow 1^-} \left| \frac{1-t}{tV} \int_0^s d\tau \int_M \left(\psi - \frac{dc_\tau}{ds} \right) \omega_{\tau,t}^n \right| \\ &= 0. \end{aligned}$$

The claim is proved. \square

By Claim 4.2 we have

$$\lim_{t \rightarrow 1^-} (\tilde{F}_\omega(\omega_{s,t}) - \tilde{F}_\omega(\omega_s)) = \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\omega_{s,t}) \leq 0.$$

and by Claim 4.3

$$\tilde{F}_\omega(\psi) = \tilde{F}_\omega(\omega_1) \geq \lim_{t \rightarrow 1^-} \tilde{F}_\omega(\omega_{1,t}) = \lim_{t \rightarrow 1^-} \tilde{F}_\omega(\omega_{0,t}).$$

Thus, \tilde{F} is uniformly bounded from below on $\mathcal{M}_X(\omega)$.

Now we prove the equality (1.1). Suppose that one of the energy functionals \tilde{E}_k and \tilde{F} is bounded from below on $\mathcal{M}_{X,k}(\omega)$, by Lemma 2.4 and the inequality (2.7) we have

$$\begin{aligned} \inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') &\geq (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{E}_{0,\omega}(\omega') + C_{\omega,X,k} \\ &\geq (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{F}_\omega(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta_X} \omega^n. \end{aligned} \quad (4.14)$$

On the other hand, for the solution $\varphi_t(t \in [0, 1))$ of (3.1) the inequality (3.4) implies that

$$\lim_{t \rightarrow 1^-} \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_t) \leq C_{\omega,X,k}.$$

Combining this with the definition of \tilde{E}_k and the equality (3.2) (4.8) we have

$$\begin{aligned} \inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') &\leq \lim_{t \rightarrow 1^-} \tilde{E}_{k,\omega}(\varphi_t) \leq (k+1) \lim_{t \rightarrow 1^-} \tilde{E}_{0,\omega}(\varphi_t) + C_{\omega,X,k} \\ &= (k+1) \left(\tilde{E}_{0,\omega}(\varphi_0) + (\tilde{I} - \tilde{J})_\omega(\varphi_0) - \int_0^1 (\tilde{I} - \tilde{J})_\omega(\varphi_\tau) d\tau \right) + C_{\omega,X,k} \\ &= (k+1) \lim_{t \rightarrow 1^-} \tilde{F}_\omega(\varphi_t) + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta_X} \omega^n \\ &= (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{F}_\omega(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta_X} \omega^n, \end{aligned} \quad (4.15)$$

where $\varphi_t(t \in [0, 1))$ is the solution of (3.1) and we have used

$$\tilde{E}_{0,\omega}(\varphi_0) + (\tilde{I} - \tilde{J})_\omega(\varphi_0) = -\frac{1}{V} \int_M u_0 e^{\theta_X} \omega^n$$

since φ_0 is a solution of (3.1) when $t = 0$. Combining (4.14)(4.15), we have

$$\inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') = (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{F}_\omega(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta_X} \omega^n.$$

The theorem is proved. \square

Following the ideas of the previous proof, we can finish Theorem 1.3:

Proof of Theorem 1.3. For any $\psi \in \mathcal{M}_X(\omega)$, we consider the solution $\varphi_{s,t}$ of the equation (4.1). Suppose that \tilde{F} is bounded from below for the solution $\varphi_{0,t}$, then $\varphi_{0,t}$ exists for all $t \in [0, 1)$ and

$$\lim_{t \rightarrow 1^-} \tilde{F}_\omega(\varphi_{0,t}) = - \int_0^1 (\tilde{I}_\omega - \tilde{J}_\omega)(\varphi_{0,\tau}) d\tau > -\infty.$$

For simplicity, we set

$$\tilde{F}_\omega^0(\varphi) = \tilde{J}_\omega(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta_X} \omega^n.$$

Then by (4.12) for any $s \in [0, 1]$ we have

$$\tilde{F}_\omega^0(\hat{\varphi}_{s,t}) - \tilde{F}_\omega^0(\hat{\varphi}_{0,t}) = - \int_0^s \frac{1-t}{tV} \int_M \left(\psi - \frac{dc_\tau}{d\tau} \right) \omega_{\tau,t}^n d\tau, \quad (4.16)$$

which implies that

$$\tilde{F}_{\omega_s}^0(\varphi_{s,t}) = \tilde{F}_\omega^0(\hat{\varphi}_{s,t}) - \tilde{F}_\omega^0(s\psi)$$

is uniformly bounded from below for any $t \in [\frac{1}{2}, 1)$ and $s \in [0, 1]$. Thus, the solution $\varphi_{s,t}$ exists for all $t \in [0, 1)$ when $s \in [0, 1]$ and

$$\lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}^0(\varphi_{s,t}) = \lim_{t \rightarrow 1^-} \tilde{F}_\omega^0(\varphi_{0,t}) - \tilde{F}_\omega^0(s\psi) > -\infty.$$

On the other hand, we have

$$(1-t)(\tilde{I}_{\omega_s}(\varphi_{s,t}) - \tilde{J}_{\omega_s}(\varphi_{s,t})) \leq \int_t^1 (\tilde{I}_{\omega_s}(\varphi_{s,t}) - \tilde{J}_{\omega_s}(\varphi_{s,t})) dt \leq - \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}^0(\varphi_{s,t})$$

and thus we have

$$\lim_{t \rightarrow 1^-} (1-t)|\varphi_{s,t}|_{C^0} = 0.$$

We can argue as (4.7) to derive

$$\lim_{t \rightarrow 1^-} \int_M e^{h_s - \varphi_{s,t}} \omega_s^n = V. \quad (4.17)$$

Thus, by the definition of \tilde{F} and (4.17) we have

$$-\infty < \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t}) = \lim_{t \rightarrow 1^-} \tilde{F}_{\omega_s}^0(\varphi_{s,t}) \leq 0,$$

and

$$\tilde{F}_\omega(\psi) = \tilde{F}_\omega(\omega_1) \geq \lim_{t \rightarrow 1^-} \tilde{F}_\omega(\omega_{1,t}) = \lim_{t \rightarrow 1^-} \tilde{F}_\omega(\omega_{0,t}),$$

where we used (4.16) in the last equality. This shows that \tilde{F} is bounded from below in the Kähler class $2\pi c_1(M)$.

Suppose that \tilde{E}_k is bounded from below for φ_t , we can see from the proof of Lemma 3.1 that \tilde{F} is also bounded from below along φ_t . Thus, \tilde{F} is bounded from below in the Kähler class $2\pi c_1(M)$ and by Theorem 1.2 \tilde{E}_k is bounded from below on $\mathcal{M}_k^+(\omega)$. The theorem is proved. \square

5 The holomorphic invariant

Recall that Tian-Zhu defined the holomorphic invariant by

$$\mathcal{F}_X(Y) = \int_M Y(h_g - \theta_X(g)) e^{\theta_X(g)} \omega_g^n, \quad Y \in \eta(M),$$

which generalized the Futaki invariant. Here $\eta(M)$ denotes the space of holomorphic vector fields on M . Let $\{\Phi(t)\}_{|t|<\infty}$ be the one-parameter subgroup of automorphisms induced by $Re(Y)$, and $\varphi(x, t)$ be the Kähler potential satisfying

$$\omega_\varphi = \Phi_t^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \varphi. \quad (5.1)$$

Differentiating (5.1), we have

$$L_{Re(Y)} \omega_\varphi = \sqrt{-1} \partial \bar{\partial} \frac{\partial \varphi}{\partial t}. \quad (5.2)$$

On the other hand, we have $L_Y \omega_\varphi = \sqrt{-1} \partial \bar{\partial} \theta_Y(\varphi)$. Thus,

$$\frac{\partial \varphi}{\partial t} = Re(\theta_Y(\varphi)) + c \quad (5.3)$$

for some constant c . Recall that u satisfies

$$\sqrt{-1} \partial \bar{\partial} u = -Ric(\omega_\varphi) + \omega_\varphi + \sqrt{-1} \partial \bar{\partial} \theta_X(\varphi).$$

Taking the interior product on both sides, we have

$$Y(u) = \Delta \theta_Y(\varphi) + \theta_Y(\varphi) + Y \theta_X(\varphi).$$

By the definition of u , we have

$$\frac{\partial u}{\partial t} = \Delta \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t} + X \left(\frac{\partial \varphi}{\partial t} \right) + c = Re(Yu) + c, \quad (5.4)$$

where we used the fact that $Y \theta_X(\varphi) = X \theta_Y(\varphi)$. Following a direct calculation, we have the lemma:

Lemma 5.1. *Let $\{\Phi(t)\}_{|t|<\infty}$ be the one-parameter subgroup of automorphisms induced by $Re(Y)$, we have*

$$\frac{d}{dt} \tilde{E}_0(\varphi) = \frac{n}{V} \mathcal{F}_X(Y), \quad (5.5)$$

where φ is given by $\Phi_t^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$.

The main result in this section is

Theorem 5.2. *Let $\{\Phi(t)\}_{|t|<\infty}$ be the one-parameter subgroup of automorphisms induced by $Re(Y)$, we have*

$$\frac{d}{dt} \tilde{E}_k(\varphi) = \frac{(k+1)n}{V} \mathcal{F}_X(Y),$$

where φ is given by $\Phi_t^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$.

Proof. By the definition of \tilde{E}_k and Lemma 5.1, it suffices to check that for any $k = 0, \dots, n$

$$\frac{d}{dt}\tilde{G}_k(\varphi) = 0. \quad (5.6)$$

Direct calculation shows that

$$\begin{aligned} \frac{d}{dt}\tilde{G}_k(\varphi) &= -\frac{1}{V}\frac{d}{dt}\int_M \sqrt{-1}\partial u \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &= -\frac{1}{V}\text{Re}\int_M 2\sqrt{-1}\partial Yu \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &\quad -\frac{1}{V}\text{Re}\int_M (k-1)\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-2} \wedge \sqrt{-1}\partial\bar{\partial}Yu \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &\quad -\frac{1}{V}\text{Re}\int_M X\theta_Y(\varphi)\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &\quad -\frac{1}{V}\text{Re}\int_M (n-k)\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k-1} \wedge \sqrt{-1}\partial\bar{\partial}\theta_Y(\varphi) \\ &= \text{Re}(I_1 + I_2 + I_3 + I_4), \end{aligned}$$

where $I_i (1 \leq i \leq 4)$ denote the integrations on the right hand side respectively. On the other hand, we have

$$\begin{aligned} 0 &= \frac{1}{V}\int_M i_Y\left(\partial u \wedge (\sqrt{-1}\partial\bar{\partial}u)^k \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k}\right) \\ &= \frac{1}{V}\int_M Yu(\sqrt{-1}\partial\bar{\partial}u)^k \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &\quad -\frac{1}{V}\int_M k\sqrt{-1}\partial u \wedge \bar{\partial} Yu \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &\quad -\frac{1}{V}\int_M (n-k)\sqrt{-1}\partial u \wedge \bar{\partial}\theta_Y(\varphi) \wedge (\sqrt{-1}\partial\bar{\partial}u)^k \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k-1} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Note that

$$\begin{aligned} J_1 &= \frac{1}{V}\int_M -\sqrt{-1}\partial Yu \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &\quad -\frac{1}{V}\int_M Yu\sqrt{-1}\partial\theta_X(\varphi) \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k} \\ &= J_{1a} + J_{1b} \end{aligned}$$

and

$$\begin{aligned} J_3 &= \frac{1}{V}\int_M -(n-k)\partial u \wedge \bar{\partial} u \wedge \sqrt{-1}\partial\bar{\partial}\theta_Y(\varphi) \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k-1} \\ &\quad +\frac{1}{V}\int_M -(n-k)\partial\theta_X(\varphi) \wedge \bar{\partial}\theta_Y(\varphi) \wedge \bar{\partial} u \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)}\omega_\varphi^{n-k-1} \\ &= J_{3a} + J_{3b}. \end{aligned}$$

Now we calculate

$$\begin{aligned}
0 &= \frac{1}{V} \int_M -i_Y \left(\partial\theta_X(\varphi) \wedge \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \right) \\
&= \frac{1}{V} \int_M -Y\theta_X(\varphi) \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \\
&\quad + \frac{1}{V} \int_M Yu \sqrt{-1}\partial\theta_X(\varphi) \wedge \bar{\partial}u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \\
&\quad + \frac{1}{V} \int_M (k-1)\partial\theta_X(\varphi) \wedge \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-2} \wedge \sqrt{-1}\bar{\partial}Yu \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \\
&\quad + \frac{1}{V} \int_M (n-k)\partial\theta_X(\varphi) \wedge \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k-1} \wedge \sqrt{-1}\bar{\partial}\theta_Y(\varphi) \\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

Note that

$$\begin{aligned}
K_3 &= \frac{1}{V} \int_M (k-1)\sqrt{-1}\partial u \wedge \bar{\partial}Yu \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \\
&\quad - \frac{1}{V} \int_M (k-1)\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge (\sqrt{-1}\partial\bar{\partial}u)^{k-2} \wedge \sqrt{-1}\partial\bar{\partial}Yu \wedge e^{\theta_X(\varphi)} \omega_\varphi^{n-k} \\
&= K_{3a} + K_{3b}.
\end{aligned}$$

Combining these equalities, we have the following relations:

$$I_1 = 2J_{1a}, \quad I_2 = K_{3b}, \quad I_3 = K_1, \quad I_4 = J_{3a}$$

and

$$J_{1b} + K_2 = 0, \quad J_{3b} + K_4 = 0, \quad J_2 + K_{3a} = J_{1a}.$$

Thus, we have

$$\sum_{i=1}^4 I_i = \sum_{i=1}^3 J_i + \sum_{i=1}^4 K_i = 0.$$

The theorem is proved. \square

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